

第3章 線形システムの時間応答

- 3.1 1次システムの時間応答
- 3.2 n次システムの時間応答

キーワード：遷移行列, 時間応答

学習目標：遷移行列の求め, 時間応答が計算できるようになる。

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3 線形システムの時間応答

3.1 1次システムの時間応答

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t), \quad x(0) = x_0 \\ y(t) &= cx(t) + du(t) \end{aligned} \quad (3.1)$$

零入力応答 $u(t) = 0$

$$\begin{aligned} \dot{x}(t) &= ax(t), \quad x(0) = x_0 \\ y(t) &= cx(t) \end{aligned} \quad (3.2)$$

(3.2)式から

$$\begin{aligned} \frac{dx(t)}{dt} &= ax(t) \Rightarrow \int \frac{dx(t)}{x(t)} = \int a dt \Rightarrow \log x(t) = at + C \\ \Rightarrow x(t) &= e^{at+C} \end{aligned}$$

初期条件 $x(0) = e^C = x_0$

$$x(t) = e^{at}x_0$$

$$y(t) = ce^{at}x_0$$

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零状態応答 $x(0) = 0$

$$\dot{x}(t) = ax(t) + bu(t) \quad (3.13)$$

$x(t) = e^{at}z(t)$ と仮定して両辺を微分すると

$$\dot{x}(t) = ae^{at}z(t) + e^{at}\dot{z}(t)$$

(3.13)式を代入する

$$\frac{ax(t) + bu(t)}{e^{at}z(t)} = \frac{ae^{at}z(t) + e^{at}\dot{z}(t)}{e^{at}z(t)}$$

$$\cancel{ae^{at}z(t)} + bu(t) = \cancel{ae^{at}z(t)} + e^{at}\dot{z}(t)$$

$$e^{at}\dot{z}(t) = bu(t)$$

$$\dot{z}(t) = e^{-at}bu(t)$$

$$z(t) = \int_0^t e^{-a\tau}bu(\tau)d\tau + \alpha$$

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$z(0) = e^{-a \times 0}x_0 = 0$ より

$$\alpha = z(0) - \int_0^0 e^{-a\tau}bu(\tau)d\tau = z(0) = 0$$

よって

$$x(t) = e^{at}z(t) = e^{at} \int_0^t e^{-a\tau}bu(\tau)d\tau = \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

$$y(t) = c \int_0^t e^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

任意の時間応答(零入力応答+零状態応答)

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

$$y(t) = ce^{at}x_0 + c \int_0^t e^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

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【例3.2】

$$\begin{cases} \dot{x}(t) = -\frac{1}{T}x(t) + \frac{K}{T}u(t), & x(0) = x_0 \\ y(t) = x(t) \end{cases}$$

$$u(t) = \begin{cases} 0 & (t < 0) \\ E & (t \geq 0) \end{cases}$$

$$y(t) = c \int_0^t e^{a(t-\tau)}bu(\tau)d\tau + du(t)$$

$$y(t) = 1 \int_0^t e^{-\frac{1}{T}(t-\tau)} \frac{K}{T} E d\tau + 0 \cdot u(t)$$

$$= \frac{KE}{T} \int_0^t e^{-\frac{1}{T}(t-\tau)} d\tau$$

$$\tilde{\tau} = t - \tau \text{ とおく} \quad \frac{d\tilde{\tau}}{d\tau} = -1 \quad \left. \begin{array}{l} \tau | 0 \rightarrow t \\ \tilde{\tau} | t \rightarrow 0 \end{array} \right\}$$

$$y(t) = -\frac{KE}{T} \int_t^0 e^{-\frac{1}{T}\tilde{\tau}} d\tilde{\tau} = \frac{KE}{T} \int_0^t e^{-\frac{1}{T}\tilde{\tau}} d\tilde{\tau}$$

$$= \frac{KE}{T} [-Te^{-\frac{1}{T}\tilde{\tau}}]_0^t = KE(1 - e^{-\frac{1}{T}t})$$

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[問題 3.1(2)]

次のシステムにおいて, $u(t) = 1 (t \geq 0)$ を加えたときの $y(t)$ を求めよ。

$$\begin{cases} \dot{x}(t) = -\frac{R}{T}x(t) + \frac{1}{T}u(t), & x(0) = 0 \\ y(t) = x(t) \end{cases}$$

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3 線形システムの時間応答

3.2 n次システムの時間応答

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (3.27)$$

遷移行列(行列指数関数)

$$e^{At} := I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots$$

例えば $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ とすると

$$\begin{aligned} e^{At} &= e^{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}t} \\ &= I + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 + \dots + \frac{t^k}{k!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k + \dots \end{aligned}$$

となる

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$$e^{A \times 0} = I$$

$$e^{A \times 0} = I + 0 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{=0} + \frac{0^2}{2!} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2}_{=0} + \dots + \frac{0^k}{k!} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k}_{=0} + \dots$$

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

$$\begin{aligned} \frac{d}{dt}e^{At} &= \frac{d}{dt} \left(I + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 + \dots + \frac{t^k}{k!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k + \dots \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \dots + \frac{t^{k-1}}{(k-1)!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k + \dots \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{=A} \left(I + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \dots + \frac{t^{k-1}}{(k-1)!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{k-1} + \dots \right) \\ &= A e^{At} = e^{At} A \end{aligned}$$

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$$A \int_0^t e^{A\tau} d\tau = e^{At} - I$$

$$A_1 A_2 = A_2 A_1 \text{ ならば } e^{A_1 t} e^{A_2 t} = e^{(A_1 + A_2)t}$$

$$e^{A t_1} e^{A t_2} = e^{A(t_1 + t_2)}$$

$$(e^{At})^{-1} = e^{-At}$$

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零入力応答 $u(t) = 0$

$$x(t) = e^{At} x_0 \quad (3.32)$$

$$y(t) = C e^{At} x_0$$

ラプラス変換による遷移行列の求め方

$$\dot{x}(t) = Ax(t)$$

$$s x(s) - x_0 = A x(s)$$

$$x(s) = (sI - A)^{-1} x_0$$

$$x(t) = \mathcal{L}^{-1}[x(s)] = \mathcal{L}^{-1}[(sI - A)^{-1}] x_0$$

(3.32) 式と比較して

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

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[例3.4](1)

$$(1) \quad A = \begin{bmatrix} 0 & 1 \\ -10 & -11 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [1 \quad 0], \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 10 & s+11 \end{bmatrix}^{-1} = \frac{1}{(s+10)(s+11)} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} \\ &= \frac{1}{s+10} K_1 + \frac{1}{s+11} K_2 \end{aligned}$$

$$\begin{aligned} K_1 &= (s+10)(sI - A)^{-1} \Big|_{s=-10} = \frac{1}{s+11} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} \Big|_{s=-10} \\ &= \frac{1}{9} \begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} K_2 &= (s+11)(sI - A)^{-1} \Big|_{s=-11} = \frac{1}{s+10} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} \Big|_{s=-11} \\ &= \frac{1}{9} \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} \end{aligned}$$

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$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 10 & s+11 \end{bmatrix}^{-1} = \frac{1}{(s+10)(s+11)} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} \\ &= \frac{1}{s+10} K_1 + \frac{1}{s+11} K_2 \\ \frac{1}{(s+10)(s+11)} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} &= \frac{1}{s+10} K_1 + \frac{1}{s+11} K_2 \\ &= (sI - A)^{-1} \end{aligned}$$

両辺に $s+10$ をかける

$$\frac{1}{s+11} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} = K_1 + \frac{s+10}{s+11} K_2 = (s+10)(sI - A)^{-1}$$

$s = -10$ を代入

$$\frac{1}{s+11} \begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix} \Big|_{s=-10} = K_1 + 0$$

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$$\begin{aligned}
 e^{At} &= \mathcal{L}^{-1} [(sI - A)^{-1}] \\
 &= \mathcal{L}^{-1} \left[\frac{1}{s+10} \frac{1}{9} \begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} + \frac{1}{s+1} \frac{1}{9} \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} \right] \\
 &= \frac{1}{9} \left(\begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} e^{-10t} + \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} e^{-t} \right) \\
 y(t) &= ce^{At}x_0 \\
 &= [1 \ 0] \frac{1}{9} \left(\begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} e^{-10t} + \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} e^{-t} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 &= \frac{1}{9} \left([-1 \ -1] e^{-10t} + [10 \ 1] e^{-t} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 &= \frac{1}{9} (-1 \times e^{-10t} + 10 \times e^{-t}) \\
 &= \frac{1}{9} (10^{-t} - e^{-10t})
 \end{aligned}$$

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[問題 3.2(1)]

線形システム

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$y(t) = Cx(t) + Du(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [1 \ 0], \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

が与えられたとき、ラプラス変換を利用して遷移行列 e^{At} を求めよ。また、零入力 $y(t)$ を求めよ。

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[MATLAB演習]

【例3.4】

ex3_4.mdl

ex3_4_parameter.m

$$A1 = [0 \ 1; -10 \ -11];$$

$$A2 = [0 \ 1; -10 \ -2];$$

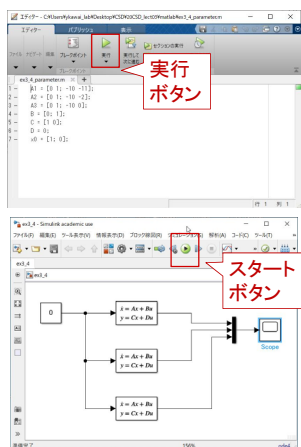
$$A3 = [0 \ 1; -10 \ 0];$$

$$B = [0; 1];$$

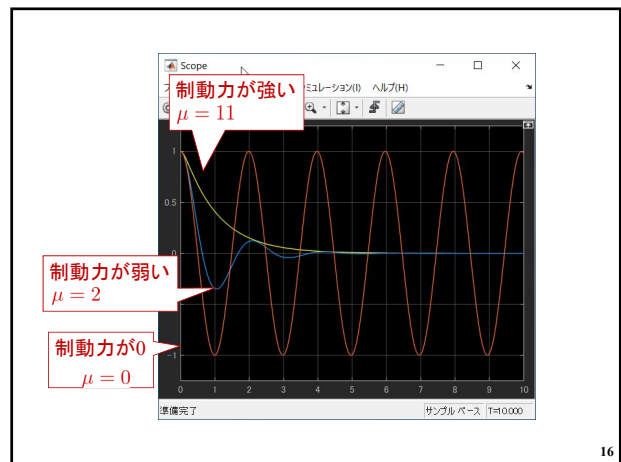
$$C = [1 \ 0];$$

$$D = 0;$$

$$x0 = [1; 0];$$



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[Scilab演習]

【例3.4】

ex3_4.zcos

ex3_4_parameter.sce

$$A1 = [0 \ 1; -10 \ -11];$$

$$A2 = [0 \ 1; -10 \ -2];$$

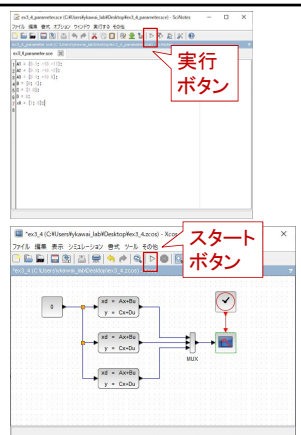
$$A3 = [0 \ 1; -10 \ 0];$$

$$B = [0; 1];$$

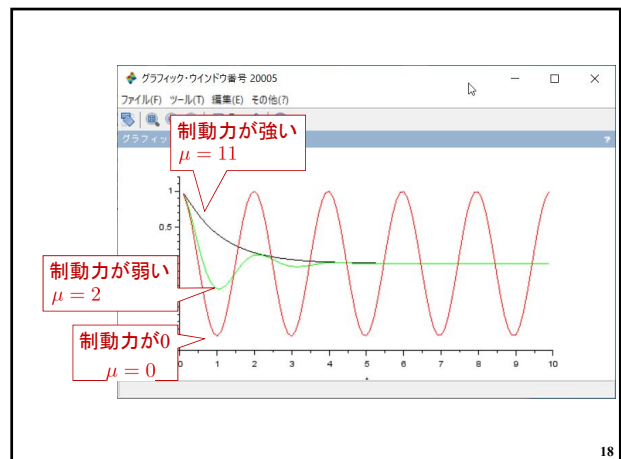
$$C = [1 \ 0];$$

$$D = 0;$$

$$x0 = [1; 0];$$



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[MATLAB演習]

3.4.1 部分分数分解

[Scilab演習]

```
clear
s=poly(0,'s');
numQ11 = s+11;
numQ12 = 1;
numQ21 = -10;
numQ22 = s;
denQ1 = s+10;
denQ2 = s+1;
k11_1 = residu(numQ11,denQ1,denQ2)
k11_2 = residu(numQ11,denQ2,denQ1)
k12_1 = residu(numQ12,denQ1,denQ2)
k12_2 = residu(numQ12,denQ2,denQ1)
k21_1 = residu(numQ21,denQ1,denQ2)
k21_2 = residu(numQ21,denQ2,denQ1)
k22_1 = residu(numQ22,denQ1,denQ2)
k22_2 = residu(numQ22,denQ2,denQ1)
K1 = [k11_1 k12_1; k21_1 k22_1]
K2 = [k11_2 k12_2; k21_2 k22_2]
```

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第3章 線形システムの時間応答

3.1 1次システムの時間応答

3.2 n次システムの時間応答

キーワード : 遷移行列, 時間応答

学習目標 : 遷移行列の求め, 時間応答が計算できるようになる。

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