

# 第3章 線形システムの時間応答

3.1 1次システムの時間応答

3.2 n次システムの時間応答

キーワード：遷移行列, 時間応答

学習目標：遷移行列の求め, 時間応答が計算できるようになる。

### 3 線形システムの時間応答

#### 3.1 1次システムの時間応答

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t), & x(0) &= x_0 \\ y(t) &= cx(t) + du(t) \end{aligned} \quad (3.1)$$

**零入力応答**  $u(t) = 0$

$$\begin{aligned} \dot{x}(t) &= ax(t), & x(0) &= x_0 \\ y(t) &= cx(t) \end{aligned} \quad (3.2)$$

(3.2) 式から

$$\begin{aligned} \frac{dx(t)}{dt} = ax(t) &\Rightarrow \int \frac{dx(t)}{x(t)} = \int a dt \Rightarrow \log x(t) = at + C \\ \Rightarrow x(t) &= e^{at+C} \end{aligned}$$

初期条件  $x(0) = e^C = x_0$

$$x(t) = e^{at} x_0$$

$$y(t) = ce^{at} x_0$$

## 零状態応答 $x(0) = 0$

$$\dot{x}(t) = ax(t) + bu(t) \quad (3.13)$$

$x(t) = e^{at}z(t)$  と仮定して両辺を微分すると

$$\dot{x}(t) = ae^{at}z(t) + e^{at}\dot{z}(t)$$

(3.13) 式を代入する

$$\frac{ax(t) + bu(t)}{e^{at}z(t)} = \frac{ae^{at}z(t) + e^{at}\dot{z}(t)}{e^{at}z(t)}$$

$$\cancel{ae^{at}z(t)} + bu(t) = \cancel{ae^{at}z(t)} + e^{at}\dot{z}(t)$$

$$e^{at}\dot{z}(t) = bu(t)$$

$$\dot{z}(t) = e^{-at}bu(t)$$

$$z(t) = \int_0^t e^{-a\tau}bu(\tau)d\tau + \alpha$$

$$z(0) = e^{-a \times 0} x_0 = 0 \text{ より}$$

$$\alpha = z(0) - \int_0^0 e^{-a\tau} bu(\tau) d\tau = z(0) = 0$$

よって

$$x(t) = e^{at} z(t) = e^{at} \int_0^t e^{-a\tau} bu(\tau) d\tau = \int_0^t e^{a(t-\tau)} bu(\tau) d\tau$$

$$y(t) = c \int_0^t e^{a(t-\tau)} bu(\tau) d\tau + du(t)$$

**任意の時間応答 (零入力応答 + 零状態応答)**

$$x(t) = e^{at} x_0 + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau$$

$$y(t) = ce^{at} x_0 + c \int_0^t e^{a(t-\tau)} bu(\tau) d\tau + du(t)$$

### 【例3.2】

$$\begin{cases} \dot{x}(t) = -\frac{1}{T}x(t) + \frac{K}{T}u(t), & x(0) = x_0 \\ y(t) = x(t) \end{cases}$$

$$u(t) = \begin{cases} 0 & (t < 0) \\ E & (t \geq 0) \end{cases}$$

$$y(t) = c \int_0^t e^{a(t-\tau)} bu(\tau) d\tau + du(t)$$

$$\begin{aligned} y(t) &= 1 \int_0^t e^{-\frac{1}{T}(t-\tau)} \frac{K}{T} E d\tau + 0 \cdot u(t) \\ &= \frac{KE}{T} \int_0^t e^{-\frac{1}{T}(t-\tau)} d\tau \end{aligned}$$

$$\tilde{\tau} = t - \tau \quad \text{とおく} \quad \frac{d\tilde{\tau}}{d\tau} = -1 \quad \begin{array}{c|c} \tau & 0 \\ \hline \tilde{\tau} & t \end{array} \begin{array}{c} \rightarrow t \\ \rightarrow 0 \end{array}$$

$$\begin{aligned} y(t) &= -\frac{KE}{T} \int_t^0 e^{-\frac{1}{T}\tilde{\tau}} d\tilde{\tau} = \frac{KE}{T} \int_0^t e^{-\frac{1}{T}\tilde{\tau}} d\tilde{\tau} \\ &= \frac{KE}{T} \left[ -Te^{-\frac{1}{T}\tilde{\tau}} \right]_0^t = KE \left( 1 - e^{-\frac{1}{T}t} \right) \end{aligned}$$

### [ 問題 3.1(2) ]

次のシステムにおいて,  $u(t) = 1$  ( $t \geq 0$ )を加えたときの  $y(t)$  を求めよ。

$$\begin{cases} \dot{x}(t) = -\frac{R}{L}x(t) + \frac{1}{L}u(t), & x(0) = 0 \\ y(t) = x(t) \end{cases}$$

### 3 線形システムの時間応答

#### 3.2 n次システムの時間応答

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t)\end{aligned}\tag{3.27}$$

#### 遷移行列(行列指数関数)

$$e^{At} := I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots$$

例えば  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  とすると

$$\begin{aligned}e^{At} &= e^{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}t} \\ &= I + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 + \cdots + \frac{t^k}{k!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k + \cdots\end{aligned}$$

となる

$$e^{A \times 0} = I$$

$$e^{A \times 0} = I + \underbrace{0 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{=0} + \underbrace{\frac{0^2}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2}_{=0} + \cdots + \underbrace{\frac{0^k}{k!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k}_{=0} + \cdots$$

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

$$\begin{aligned} \frac{d}{dt} e^{At} &= \frac{d}{dt} \left( I + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 + \cdots + \frac{t^k}{k!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k + \cdots \right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 + \cdots + \frac{t^{k-1}}{(k-1)!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k + \cdots \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{=A} \underbrace{\left( I + t \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \cdots + \frac{t^{k-1}}{(k-1)!} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{k-1} + \cdots \right)}_{=e^{At}} \end{aligned}$$



$$A \int_0^t e^{A\tau} d\tau = e^{At} - I$$

$$A_1 A_2 = A_2 A_1 \text{ ならば } e^{A_1 t} e^{A_2 t} = e^{(A_1 + A_2)t}$$

$$e^{A t_1} e^{A t_2} = e^{A(t_1 + t_2)}$$

$$(e^{At})^{-1} = e^{-At}$$

零入力応答  $u(t) = 0$

$$x(t) = e^{At} x_0 \quad (3.32)$$

$$y(t) = C e^{At} x_0$$

ラプラス変換による遷移行列の求め方

$$\dot{x}(t) = Ax(t)$$

$$sx(s) - x_0 = Ax(s)$$

$$x(s) = (sI - A)^{-1} x_0$$

$$x(t) = \mathcal{L}^{-1}[x(s)] = \mathcal{L}^{-1} [(sI - A)^{-1}] x_0$$

(3.32) 式と比較して

$$e^{At} = \mathcal{L}^{-1} [(sI - A)^{-1}]$$

### [例3.4](1)

$$(1) \quad A = \begin{bmatrix} 0 & 1 \\ -10 & -11 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [1 \quad 0], \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 10 & s + 11 \end{bmatrix}^{-1} = \frac{1}{(s + 10)(s + 1)} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} \\ &= \frac{1}{s + 10} K_1 + \frac{1}{s + 1} K_2 \end{aligned}$$

$$\begin{aligned} K_1 &= (s + 10)(sI - A)^{-1} \Big|_{s=-10} = \frac{1}{s + 1} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} \Big|_{s=-10} \\ &= \frac{1}{9} \begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} K_2 &= (s + 1)(sI - A)^{-1} \Big|_{s=-1} = \frac{1}{s + 10} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} \Big|_{s=-1} \\ &= \frac{1}{9} \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 10 & s + 11 \end{bmatrix}^{-1} = \frac{1}{(s + 10)(s + 1)} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} \\
 &= \frac{1}{s + 10} K_1 + \frac{1}{s + 1} K_2
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{(s + 10)(s + 1)} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} &= \frac{1}{s + 10} K_1 + \frac{1}{s + 1} K_2 \\
 \hline
 &= (sI - A)^{-1}
 \end{aligned}$$

両辺に  $s + 10$  をかける

$$\begin{aligned}
 \frac{1}{s + 1} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} &= K_1 + \frac{s + 10}{s + 1} K_2 \\
 \hline
 &= (s + 10)(sI - A)^{-1}
 \end{aligned}$$

$s = -10$  を代入

$$\frac{1}{s + 1} \begin{bmatrix} s + 11 & 1 \\ -10 & s \end{bmatrix} \Big|_{s=-10} = K_1 + 0$$

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1} [(sI - A)^{-1}] \\
&= \mathcal{L}^{-1} \left[ \frac{1}{s+10} \frac{1}{9} \begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} + \frac{1}{s+1} \frac{1}{9} \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} \right] \\
&= \frac{1}{9} \left( \begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} e^{-10t} + \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} e^{-t} \right)
\end{aligned}$$

$$\begin{aligned}
y(t) &= ce^{At} x_0 \\
&= [1 \quad 0] \frac{1}{9} \left( \begin{bmatrix} -1 & -1 \\ 10 & 10 \end{bmatrix} e^{-10t} + \begin{bmatrix} 10 & 1 \\ -10 & -1 \end{bmatrix} e^{-t} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{9} \left( [-1 \quad -1] e^{-10t} + [10 \quad 1] e^{-t} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{9} \left( -1 \times e^{-10t} + 10 \times e^{-t} \right) \\
&= \frac{1}{9} \left( 10e^{-t} - e^{-10t} \right)
\end{aligned}$$

## [ 問題 3.2(1) ]

### 線形システム

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

$$y(t) = Cx(t) + Du(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = [1 \quad 0], \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

が与えられたとき、ラプラス変換を利用して遷移行列  $e^{At}$  を求めよ。また、零入力  $y(t)$  を求めよ。

# [ MATLAB演習 ]

## 【例3.4】

ex3\_4.mdl

ex3\_4\_parameter.m

$$A1 = [0 \ 1; -10 \ -11];$$

$$A2 = [0 \ 1; -10 \ -2];$$

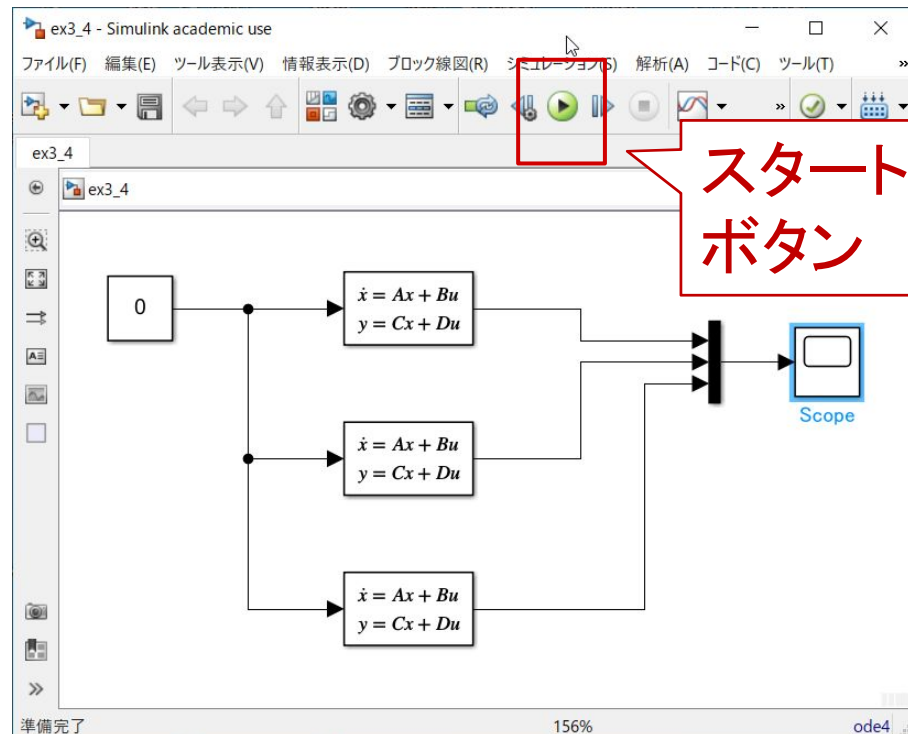
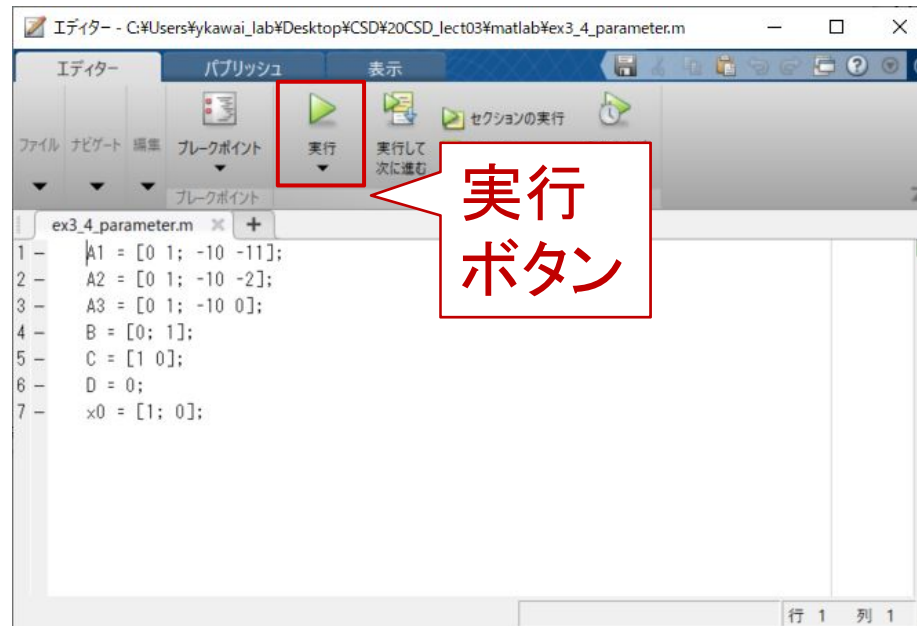
$$A3 = [0 \ 1; -10 \ 0];$$

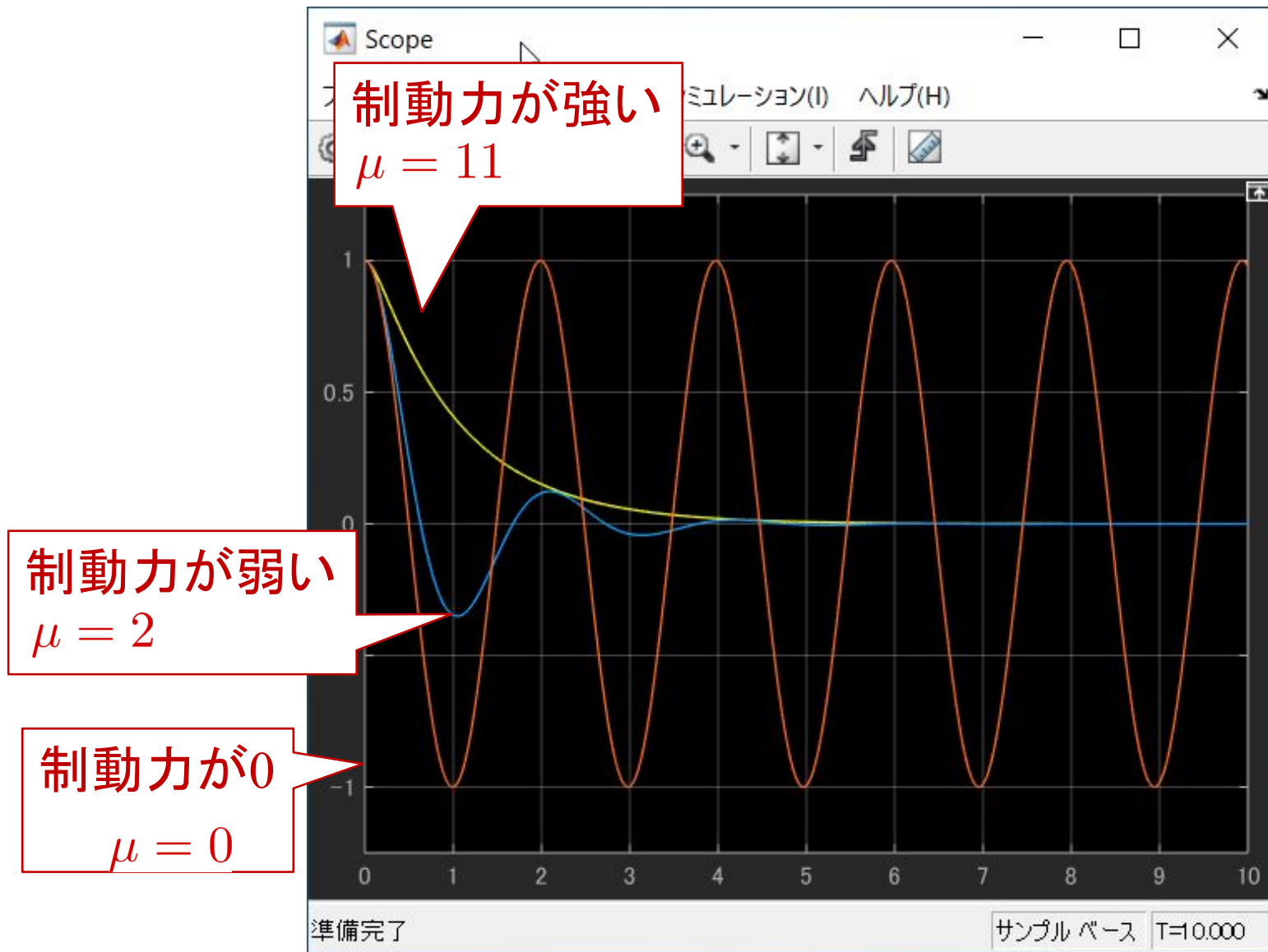
$$B = [0; 1];$$

$$C = [1 \ 0];$$

$$D = 0;$$

$$x0 = [1; 0];$$







# [ Scilab演習 ]

## 【例3.4】

ex3\_4.zcos

ex3\_4\_parameter.sce

$$A1 = [0 \ 1; -10 \ -11];$$

$$A2 = [0 \ 1; -10 \ -2];$$

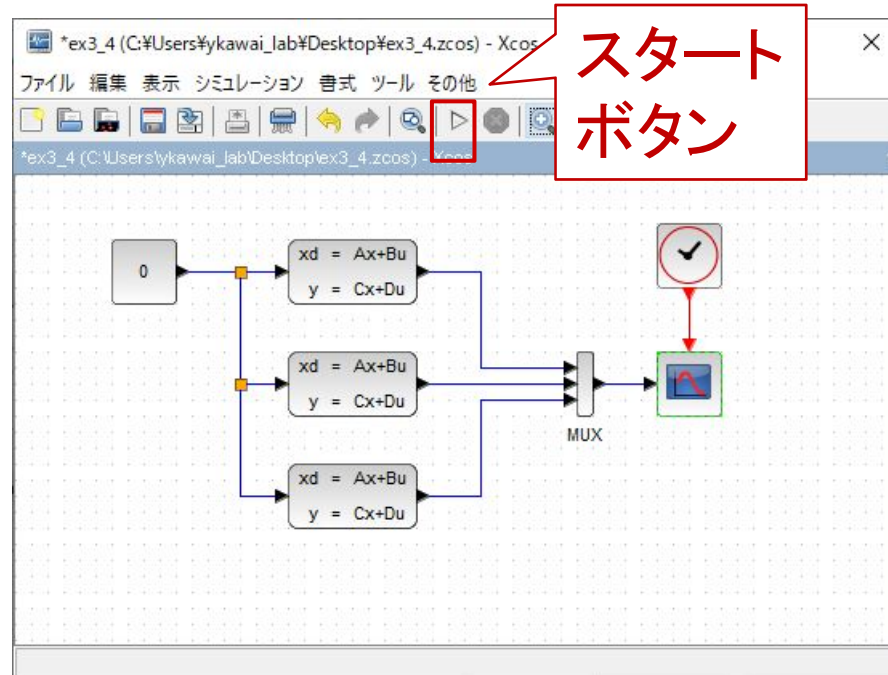
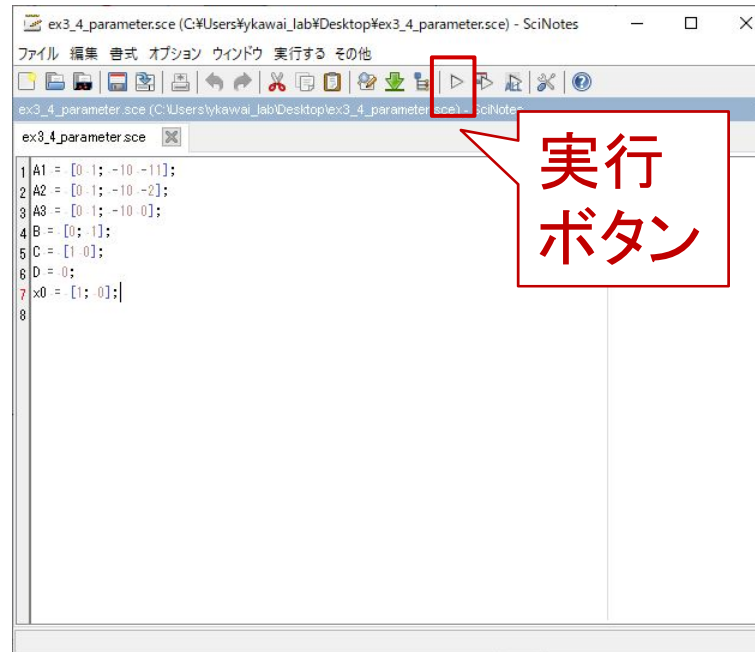
$$A3 = [0 \ 1; -10 \ 0];$$

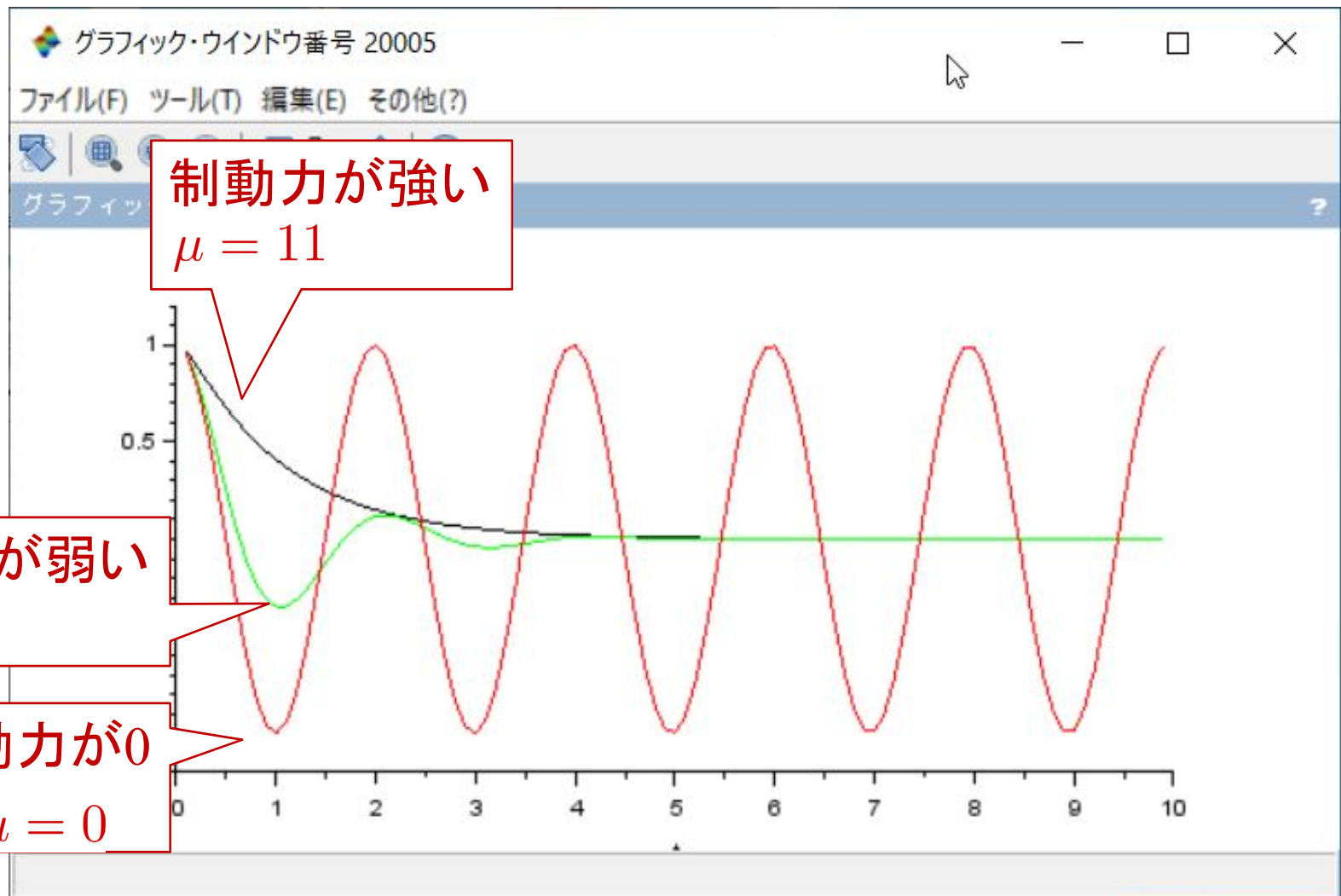
$$B = [0; 1];$$

$$C = [1 \ 0];$$

$$D = 0;$$

$$x0 = [1; 0];$$





## [ MATLAB演習 ]

### 3.4.1 部分分数分解

## [ Scilab演習 ]

```
clear
s=poly(0,'s');
numQ11 = s+11;
numQ12 = 1;
numQ21 = -10;
numQ22 = s;
denQ1 = s+10;
denQ2 = s+1;
k11_1 = residu(numQ11,denQ1,denQ2)
k11_2 = residu(numQ11,denQ2,denQ1)
k12_1 = residu(numQ12,denQ1,denQ2)
k12_2 = residu(numQ12,denQ2,denQ1)
k21_1 = residu(numQ21,denQ1,denQ2)
k21_2 = residu(numQ21,denQ2,denQ1)
k22_1 = residu(numQ22,denQ1,denQ2)
k22_2 = residu(numQ22,denQ2,denQ1)
K1 = [k11_1 k12_1; k21_1 k22_1]
K2 = [k11_2 k12_2; k21_2 k22_2]
```

## 第3章 線形システムの時間応答

3.1 1次システムの時間応答

3.2 n次システムの時間応答

キーワード：遷移行列, 時間応答

学習目標：遷移行列の求め, 時間応答が計算できるようになる。