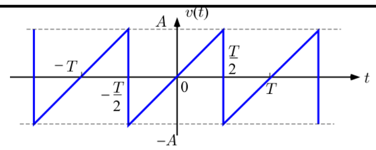
[問題30.1] (2)

$$\omega = \frac{2\pi}{T}$$



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

【解法1】奇関数とは知らずに求める

a_0 を求める

$$a_0 = \frac{1}{T}\int_0^T v(t)\ dt = \frac{1}{T}\left(\int_0^{\frac{T}{2}} \frac{2A}{T}t\ dt + \int_{\frac{T}{2}}^{T}\left(\frac{2A}{T}t - 2A\right)\ dt\right)$$

$$= \frac{1}{T}\left(\frac{2A}{T}\left[\frac{1}{2}t^2\right]_0^{\frac{T}{2}} + \frac{2A}{T}\left[\frac{1}{2}t^2\right]_{\frac{T}{2}}^{T} - 2A[t]_{\frac{T}{2}}^{T}\right)$$

$$= \frac{1}{T}\left(\frac{2A}{T}\left(\frac{1}{8}T^2 + \frac{1}{2}T^2 - \frac{1}{8}T^2\right) - 2A\frac{T}{2}\right)$$

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$$= \frac{1}{T}\left(\frac{2A}{T}\left(\frac{1}{8}T^2 + \frac{1}{2}T^2 - \frac{1}{8}T^2\right) - 2A\frac{T}{2}\right)$$

$$= \frac{1}{T}\left(\frac{2A}{T}\frac{1}{2}T^2 - 2A\frac{T}{2}\right)$$

$$= \frac{1}{T}(AT - AT)$$

$$= 0$$

別解

$$a_0 = \frac{1}{T}\int_0^T v(t)\ dt = \frac{1}{T}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T}t\ dt\right)$$

$$= \frac{2A}{T^2}\left[\frac{1}{2}t^2\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= 0$$

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$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega t\ dt$$

$$= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T}t\cos n\omega t\ dt$$

$$= \frac{4A}{T^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} t\underbrace{\left(\frac{1}{n\omega}\sin n\omega t\right)'}_{=\cos n\omega t}\ dt$$

$$= \frac{4A}{T^2}\left\{\left[t\left(\frac{1}{n\omega}\sin n\omega t\right)\right]_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{n\omega}\sin n\omega t\ dt\right\}$$ 部分積分

$$= \frac{4A}{n\omega T^2}\left\{[t\sin n\omega t]_0^{\frac{T}{2}} - \underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin n\omega t\ dt}_{= -\left[\frac{1}{n\omega}\cos n\omega t\right]_{-\frac{T}{2}}^{\frac{T}{2}}}\right\}$$

$$= \frac{4A}{n\omega T^2}\left\{\frac{T}{2}\sin n\omega\frac{T}{2} - 0 + \frac{1}{n\omega}\underbrace{\left(\cos n\omega\frac{T}{2} - \cos n\omega\left(-\frac{T}{2}\right)\right)}_{=0}\right\}$$

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$$= \frac{4A}{n\omega T^2}\left\{\frac{T}{2}\sin n\omega\frac{T}{2} - 0 + \frac{1}{n\omega}\underbrace{\left(\cos n\omega\frac{T}{2} - \cos n\omega\left(-\frac{T}{2}\right)\right)}_{=0}\right\}$$

$$= \frac{4A}{n\omega T^2}\frac{T}{2}\sin n\omega\frac{T}{2}$$

$\omega T = 2\pi$ より

$$a_n = \frac{4A}{n\omega T^2}\frac{T}{2}\underbrace{\sin n\pi}_{=0} = 0$$

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$$b_n = \frac{2}{T}\int_0^T v(t)\sin n\omega t\ dt$$

$$= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T}t\sin n\omega t\ dt$$

$$= \frac{4A}{T^2}\int_{-\frac{T}{2}}^{\frac{T}{2}} t\underbrace{\left(-\frac{1}{n\omega}\cos n\omega t\right)'}_{=\sin n\omega t}\ dt$$

$$= \frac{4A}{T^2}\left\{\left[t\left(-\frac{1}{n\omega}\cos n\omega t\right)\right]_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} -\frac{1}{n\omega}\cos n\omega t\ dt\right\}$$

$$= -\frac{4A}{n\omega T^2}\left\{[t\cos n\omega t]_{-\frac{T}{2}}^{\frac{T}{2}} - \underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos n\omega t\ dt}_{= \left[\frac{1}{n\omega}\sin n\omega t\right]_{-\frac{T}{2}}^{\frac{T}{2}}}\right\}$$

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$$= -\frac{4A}{n\omega T^2}\left\{\frac{T}{2}\cos n\omega\frac{T}{2} - \left(-\frac{T}{2}\right)\cos\left(-n\omega\frac{T}{2}\right) - \frac{1}{n\omega}\left(\sin n\omega\frac{T}{2} - \sin\left(-n\omega\frac{T}{2}\right)\right)\right\}$$

$$= -\frac{4A}{n\omega T^2}\left\{2\frac{T}{2}\cos n\omega\frac{T}{2} - \frac{2}{n\omega}\sin n\omega\frac{T}{2}\right\}$$

$$= -\frac{8A}{n\omega T^2}\left\{\frac{T}{2}\cos n\omega\frac{T}{2} - \frac{1}{n\omega}\sin n\omega\frac{T}{2}\right\}$$

$\omega T = 2\pi$ より

$$b_n = -\frac{8A}{n2\pi T}\left\{\frac{T}{2}\cos n\pi - \underbrace{\frac{1}{n\omega}(\sin n\pi)}_{=0}\right\}$$

$$= -\frac{2A}{n\pi}\cos n\pi$$

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$$b_1 = -\frac{2A}{\pi}\cos\pi = \frac{2A}{\pi}$$

$$b_2 = -\frac{2A}{2\pi}\cos 2\pi = -\frac{2A}{2\pi}$$

$$b_3 = -\frac{2A}{3\pi}\cos 3\pi = \frac{2A}{3\pi}$$

よって

$$v(t) = \frac{2A}{\pi}\left(\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right)$$

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【解法2】奇関数を使う

奇関数より $a_0 = 0,\ a_n = 0$

b_n を求める

(A1)

$$b_n = \frac{2}{T}\int_0^T v(t)\sin n\omega t\ dt$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{2}} \frac{2A}{T}t\sin n\omega t\ dt + \int_{\frac{T}{2}}^T \left(\frac{2A}{T}t - 2A\right)\sin n\omega t\ dt\right)$$

(A2)半周期を2倍

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} v(t)\sin n\omega t\ dt\ = \frac{4}{T}\int_0^{\frac{T}{2}} \frac{2A}{T}t\sin n\omega t\ dt$$

(A3)周期を $-\frac{T}{2} \sim \frac{T}{2}$ と考える

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)\sin n\omega t\ dt\ = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T}t\sin n\omega t\ dt$$

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b_n を求める

(A2)で解く

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} \frac{2A}{T}t\sin n\omega t\ dt$$

$$= \frac{8A}{T^2}\int_0^{\frac{T}{2}} t\underbrace{\left(-\frac{1}{n\omega}\cos n\omega t\right)'}_{=\sin n\omega t}\ dt$$

部分積分を用いる

$$b_n = \frac{8A}{T^2}\left\{\left[t\left(-\frac{1}{n\omega}\cos n\omega t\right)\right]_0^{\frac{T}{2}} - \int_0^{\frac{T}{2}} -\frac{1}{n\omega}\cos n\omega t\ dt\right\}$$

$$= -\frac{8A}{n\omega T^2}\left\{[t\cos n\omega t]_0^{\frac{T}{2}} - \underbrace{\int_0^{\frac{T}{2}}\cos n\omega t\ dt}_{=\left[\frac{1}{n\omega}\sin n\omega t\right]_0^{\frac{T}{2}}}\right\}$$

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$$= -\frac{8A}{n\omega T^2}\left\{\frac{T}{2}\cos n\omega\frac{T}{2} - 0 - \frac{1}{n\omega}\left(\sin n\omega\frac{T}{2} - \sin 0\right)\right\}$$

$\omega T = 2\pi$ より

$$b_n = -\frac{8A}{n2\pi T}\left\{\frac{T}{2}\cos n\pi - \underbrace{\frac{1}{n\omega}(\sin n\pi)}_{=0}\right\}$$

$$= -\frac{2A}{n\pi}\cos n\pi$$

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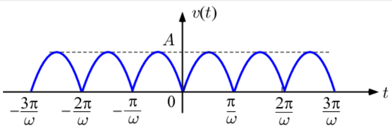
[問題30.1](3)

$$\omega = \frac{2\pi}{T}$$



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\omega t + \sum_{n=1}^{\infty} b_n\sin n\omega t$$

偶関数より $b_n = 0$

a_0 を求める

(解法1)

$$a_0 = \frac{1}{T}\int_0^T v(t)\ dt\ = \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\sin\omega t\ dt + \int_{\frac{T}{2}}^T (-A\sin\omega t)\ dt\right)$$

(解法2)半周期を2倍

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} v(t)\ dt\ = \frac{2}{T}\int_0^{\frac{T}{2}} A\sin\omega t$$

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(解法1)で解く

$$a_0 = \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\sin\omega t\ dt + \int_{\frac{T}{2}}^T (-A\sin\omega t)\ dt\right)$$

$$= \frac{A}{T}\left(\left[-\frac{1}{\omega}\cos\omega t\right]_0^{\frac{T}{2}} + \left[\frac{1}{\omega}\cos\omega t\right]_{\frac{T}{2}}^T\right)$$

$$= \frac{A}{\omega T}\left([-\cos\omega t]_0^{\frac{T}{2}} + [\cos\omega t]_{\frac{T}{2}}^T\right)$$

$$= \frac{A}{\omega T}\left(-\cos\frac{\omega T}{2} + \cos 0 + \cos\omega T - \cos\frac{\omega T}{2}\right)$$

$$= \frac{A}{2\pi}(-\cos\pi + 1 + \cos 2\pi - \cos\pi)$$

$$= \frac{A}{2\pi}(1 + 1 + 1 + 1)$$

$$= \frac{2A}{\pi}$$

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（解法2）で解く

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} A\sin\omega t$$
$$= \frac{2A}{T}\left(\left[-\frac{1}{\omega}\cos\omega t\right]_0^{\frac{T}{2}}\right)$$
$$= \frac{2A}{\omega T}\left([-\cos\omega t]_0^{\frac{T}{2}}\right)$$
$$= \frac{2A}{\omega T}\left(-\cos\frac{\omega T}{2}+\cos 0\right)$$
$$= \frac{2A}{2\pi}(-\cos\pi+1)$$
$$= \frac{2A}{2\pi}(1+1)$$
$$= \frac{2A}{\pi}$$

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（解法2）で解く

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} v(t)\cos n\omega t\ dt$$
$$= \frac{4}{T}\int_0^{\frac{T}{2}} A\sin\omega t\cos n\omega t\ dt$$

$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta)$ を用いて

$$a_n = \frac{2A}{T}\int_0^{\frac{T}{2}}\left(\sin(n+1)\omega t - \sin(n-1)\omega t\right)\ dt$$
$$= \frac{2A}{T}\left(\frac{1}{(n+1)\omega}[-\cos(n+1)\omega t]_0^{\frac{T}{2}} - \frac{1}{(n-1)\omega}[-\cos(n-1)\omega t]_0^{\frac{T}{2}}\right)$$
$$= \frac{2A}{\omega T}\left(-\frac{1}{n+1}[\cos(n+1)\omega t]_0^{\frac{T}{2}} + \frac{1}{n-1}[\cos(n-1)\omega t]_0^{\frac{T}{2}}\right)$$
$$= \frac{2A}{\omega T}\left\{-\frac{1}{n+1}\left(\cos(n+1)\frac{\omega T}{2}-\cos 0\right)+\frac{1}{n-1}\left(\cos(n-1)\frac{\omega T}{2}-\cos 0\right)\right\}$$

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$\omega = \frac{2\pi}{T}$ を用いて

$$a_n = \frac{2A}{2\pi}\left\{-\frac{1}{n+1}(\cos(n+1)\pi-1)+\frac{1}{n-1}(\cos(n-1)\pi-1)\right\}$$

$$a_1 = \frac{A}{\pi}\left\{-\frac{1}{2}(\cos 2\pi-1)+\frac{1}{1-1}(\cos 0-1)\right\} = 0$$
$$a_2 = \frac{A}{\pi}\left\{-\frac{1}{3}(\cos 3\pi-1)+\frac{1}{1}(\cos\pi-1)\right\} = \frac{A}{\pi}\left\{-\frac{1}{3}(-2)-2\right\} = \frac{A}{\pi}\left(-\frac{4}{3}\right)$$
$$a_3 = \frac{A}{\pi}\left\{-\frac{1}{4}(\cos 4\pi-1)+\frac{1}{2}(\cos 2\pi-1)\right\} = 0$$
$$a_4 = \frac{A}{\pi}\left\{-\frac{1}{5}(\cos 5\pi-1)+\frac{1}{3}(\cos 3\pi-1)\right\} = \frac{A}{\pi}\left\{-\frac{1}{5}(-2)+\frac{1}{3}(-2)\right\}$$
$$= \frac{A}{\pi}\left(\frac{2}{5}-\frac{2}{3}\right) = \frac{A}{\pi}\left(\frac{6-10}{15}\right) = \frac{A}{\pi}\left(\frac{-4}{15}\right)$$

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よって

$$v(t) = \frac{4A}{\pi}\left(\frac{1}{2}-\frac{1}{3}\cos 2\omega t-\frac{1}{15}\cos 4\omega t\cdots\right)$$

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